CENGAGE

MATHEMATICS

for Elementary School Teachers 7e

BASSAREAR • MOSS

Four Steps for Solving Problems

Understanding the Problem

Questions that can be useful to ask:

1. Do you understand what the problem is asking for?
2. Can you state the problem in your own words, that is, paraphrase the problem?
3. Have you used all the given information?
4. Can you solve a part of the problem?

Actions that can be helpful:

1. Reread the problem carefully. (Often it helps to reread a problem a few times.)
2. Try to use the given information to deduce more information.
3. Plug in some numbers to make the problem more concrete, more real.

Devising a Plan

Several common strategies:

1. Represent the problem with a diagram (carefully drawn and labeled).
   Check to see if you used (the relevant) given information. Does the diagram "fit" the problem?
2. Guess–check–revise (vs. "grope and hope"). Keep track of "guesses" with a table.
3. Make an estimate. The estimate often serves as a useful "check." A solution plan often comes from the estimation process.
4. Make a table (sometimes the key comes from adding a new column).
5. Look for patterns—in the problem or in your guesses.
6. Be systematic.
7. Look to see if the problem is similar to one already solved.
8. If the problem has "ugly" numbers, you may "see" the problem better by substituting "nice" numbers and then thinking about the problem.
9. Break the problem down into a sequence of simpler "bite-size" problems.
10. Act it out.

Carrying Out the Plan

1. Are you keeping the problem meaningful or are you just "groping and hoping?" On each step ask what the numbers mean. Label your work.
2. Are you bogged down? Do you need to try another strategy?

Looking Back

1. Does your answer make sense? Is the answer reasonable? Is the answer close to your estimate, if you made one?
2. Does your answer work when you check it with the given information? (Note that checking the procedure checks the computation but not the solution.)
3. Can you use a different method to solve the problem?

This sheet is meant to serve as a starting point. The number of strategies that help the problem-solving process are almost endless and vary according to each person's strengths and preferences.

After you solve a problem that was challenging for you or after you find that your answer was wrong, stop and reflect. Can you describe what you did that got you unstuck or things you did that helped you to solve the problem? If your answer was wrong, can you see what you might have done? *It is the depth of these reflections that connects to your increased ability to solve problems.*

The Eight Mathematical Practices of the Common Core State Standards

MP 1 Make sense of problems and persevere in solving them.

- Mathematically proficient students look for a place to get started. That is often the hardest part—where do I start?
- They think, try something, assess if it is helpful, and then continue if it was useful—or try another plan.
- If they recognize this problem as similar to one they have solved, they adapt what they used in that similar problem.
- They simplify the problem—making the numbers smaller or simpler.
- If they are heading down a path that is not solving the problem, they are aware of it and try something different.
- Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves "Does this make sense?"

MP 2 Reason abstractly and quantitatively.

- Mathematically proficient students make sense of numbers and their context within a problem.
- They are able to "decontextualize" a problem by representing it with numbers and symbols, that abstracts away from the context.
- They are also able to "contextualize" the symbolic manipulations by pausing to go back to the context when needed.
- They are able to make sense of mathematical ideas.

MP 3 Construct viable arguments and critique the reasoning of others.

- Mathematically proficient students are able to use definitions and previous knowledge to communicate their understanding.
- They are able to build a logical progression of their ideas.
- They are able to use counterexamples to make an argument.
- Elementary students can make sense of math and communicate by using objects, drawings, diagrams, or actions.
- They can listen to the reasoning of others and ask useful questions to clarify.

MP 4 Model with mathematics.

- Mathematically proficient students can *apply* the mathematics they know to solve problems arising in everyday life, society, other subjects, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation.
- They are able to identify important information in a real-life situation.
- They are able to use objects, pictures, tables, graphs, and equations to explore problems.
- They routinely interpret their model and improve the model when necessary.

MP 5 Use appropriate tools strategically.

- Mathematically proficient students consider a variety of available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, or a spreadsheet.
- They are able to make good decisions about when each of these tools might be helpful.
- They are able to use a variety of tools to explore and deepen their understanding of ideas.

MP 6 Attend to precision.

- Mathematically proficient students communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning.
- They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.
- They are careful about specifying units of measure.
- They calculate accurately and efficiently; they express numerical answers with a degree of precision appropriate for the problem context.

MP 7 Look for and make use of structure.

- Mathematically proficient students look closely to discern a pattern or structure.
- They use these familiar and known structures to see things in different ways and extend their understanding.
- Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have.
- Later, students will see that 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property.

MP 8 Look for and express regularity in repeated reasoning.

- Mathematically proficient students notice if calculations are repeated, and look for both general methods and shortcuts.
- They continue to look at their process and evaluate their results.
- They develop new methods by generalizing patterns.

The entire Common Core State Standards for mathematics can be viewed at www.corestandards.org/math/

Seventh Edition

Mathematics for Elementary School Teachers

TOM BASSAREAR
Keene State College

MEG MOSS
Western Governors University

CENGAGE

Australia • Brazil • Mexico • Singapore • United Kingdom • United States

***Mathematics for Elementary School Teachers*, Seventh Edition**
Tom Bassarear, Meg Moss

Product Director: Mark Santee

Product Manager: Spencer Arritt

Product Assistant: Amanda Rose

Marketing Manager: Shannon Hawkins

Content Manager: Adrian Daniel

Learning Designer: Mona Zeftel

IP Analyst: Reba Frederics

IP Project Manager: Carly Belcher

Production Service/Compositor: SPi Global

Senior Designer: Diana Graham

Cover Designer: Vernon Boes

Cover Image: Pavelia/Shutterstock.com

Manufacturing Planner: Becky Cross

Library of Congress Control Number: 2018959972

Student Edition: ISBN: 978-1-337-62996-6
Loose-leaf Edition: ISBN: 978-0-357-04387-5

Cengage
20 Channel Center Street
Boston, MA 02210
USA

Cengage is a leading provider of customized learning solutions with employees residing in nearly 40 different countries and sales in more than 125 countries around the world. Find your local representative at **www.cengage.com.**

Cengage products are represented in Canada by Nelson Education, Ltd.

To learn more about Cengage platforms and services, register or access your online learning solution, or purchase materials for your course, visit **www.cengage.com.**

Printed in the United States of America
Print Number: 01 Print Year: 2018

BRIEF CONTENTS

CONTENTS

8 Geometry as Shape 435

9 Geometry as Measurement 515

10 Geometry as Transforming Shapes 567

ABOUT THE AUTHORS

Tom Bassarear

I taught the Mathematics for Elementary School Teachers course for more than 20 years, and in that time, I learned as much from my students as they learned from me. This text was inspired by my students and reflects one of the most important things we have taught one another: that building an understanding of mathematics is an active, exploratory process, and ultimately a rewarding, pleasurable one. My own experience with elementary schoolchildren and my two children, Emily and Josh, has convinced me that young children naturally seek to make sense of the world they live in and for a variety of reasons many people slowly lose that curiosity over time. My hope is that this book will engage your curiosity about mathematics once again. I am fully retired now and enjoying my first grandchild, and I am so excited about the changes that Meg has made in this edition.

Meg Moss

I am excited and honored to be working with Tom Bassarear on this book. I began teaching Mathematics for Elementary Teachers over 20 years ago. I immediately began seeking advice from others who had taught the course and volunteered in elementary classrooms to learn more. Teaching these courses has deepened my mathematical understanding as well as my understanding of how people learn math. Helping future elementary school teachers to truly understand mathematics and see the beauty in mathematics is very rewarding, and I know that all of this will have a major positive impact on their future students. I appreciate you sharing this journey with me!

A MESSAGE FROM TOM BASSAREAR

We want to introduce ourselves. I began my teaching career in 1973 (I know, "long time ago"), and I just loved teaching. However, I was stunned that so many otherwise intelligent high school students could be so poor at math. Nine years later, after a stint teaching math in Nepal in the Peace Corps, I went to graduate school to get a better understanding of why so many adults had difficulty with math. Four years later, I had a much better understanding, and I became an assistant professor at Keene State College in New Hampshire where I worked until my retirement in 2016.

The idea to write the book came to me in 1993 when a textbook representative asked me if I was happy with the book I was using. I said, "No, but I'm not happy with any of the books." When the rep asked me why and what I was looking for, I explained in detail. Her response was to ask me if I'd consider writing a book. I said, "I don't think what I want would sell because it's out of the mainstream." The rep replied that she felt the time was ripe for a book that was "outside the box." Three months later, I sent out a book proposal. Houghton Mifflin agreed that the time was right for a new kind of book for future teachers of elementary school mathematics, because there had been such a strong positive reaction to the groundbreaking standards by the National Council of Teachers of Mathematics in 1989. Houghton let me write the books I wanted—the textbook and the *Explorations Manual*, which had problems designed to develop stronger problem-solving and reasoning skills. I wrote the textbook in a conversational style (and have received emails from students telling me this was the first math textbook they could actually read). I also use the term "Investigations" for the examples in the book because I had learned that people retain (own) the knowledge they learn in class much better if they are more actively involved as opposed to simply listening to the teacher lecture. After the fourth edition, Houghton Mifflin sold their college textbook division to Cengage, and I have been delighted with all the support from Cengage and the new features that have been added since then.

After I finished the fifth edition, I told my editors that I wanted to bring a new author onto the book. We asked interested teachers to submit their ideas for a new edition, and Meg's work stood out heads and shoulders above the others. Meg and I worked closely together on the sixth edition. I was so impressed by her dedication and integrity in wanting to make mathematics meaningful and accessible to all students. Other college teachers were also impressed because the sixth edition was well received. I am now fully retired and enjoying my first grandchild, and I am so excited at the changes that Meg has made in this edition.

NEW TO THE SEVENTH EDITION!

We are excited about this edition and the changes made. We maintained the same philosophy in this edition and expanded the ideas of "owning" math knowledge and actively engaged learners. The main changes to the text are:

- More colors have been added to the book for visual appeal as well as to make the concepts clearer for students.
- More Investigations have been added, especially ones that help develop concepts through a concrete, pictorial, and then abstract pathway. This framework for learning math concepts was developed by Jerome Bruner and has shown success in Singapore.
- The information in the sidebars have been incorporated into the text as students tend to find sidebars distracting.
- The WebAssign course now includes explorations, conceptual questions, and classroom videos. We plan yearly updates and welcome your feedback.
- The Explorations Manual is now available to use for free on both the instructor's and student's websites.
- Because students tended to get bogged down in Chapter 1, it has been shortened, with the investigations integrated into the related content chapter. The concept of developing a "mathematical mindset" has been added to this chapter. The section on Sets, which was previously in Chapter 2, has been moved to Chapter 1 as a way to get started.
- Chapters 2–4 have been reorganized with the operations as the framework instead of number sets.
    - Chapter 2 goes in depth into each of the number sets from whole numbers to real numbers.
    - Chapter 3 develops the concepts of addition and subtraction, showing both the relationships between these two operations and how these operations are similar to each of the different number sets. This enables students to better see these connections.
    - Chapter 4 develops the concepts of multiplication and division, again showing the connections between the operations and the number sets.
- Chapter 7 has been reorganized using the National Council of Teachers of Mathematics Principles and Standards of School Mathematics as a framework. The first three sections develop concepts around statistics. The fourth section focuses on how probability and counting techniques are connected. The examples and data sets in Chapter 7 have been updated.
- Number Theory concepts are integrated where they are applicable, such as connecting the greatest common factor with simplifying fractions.
- The pages from elementary school math books have all been updated.
- All of the chapters have been revised and reviewed with attention to detail, brevity, and clarity, as well as ensuring that all examples are current.
- The Geometry chapters (8–10) have been enhanced with added colors and Investigations dedicated to developing concepts.

Chapter 1 Foundations for Learning Mathematics

This chapter introduces the NCTM standards as well as the Mathematical Practices of the Common Core State Standards. The idea of a mathematical growth mindset has been added. The first section of this chapter lays the groundwork for seeing the standards "in action" and to understanding mathematics as a connected and interesting subject. We have integrated some of the problem-solving investigations into the other chapters where the content was more relevant. The concepts of sets has been moved into Chapter 1 to lay the framework of thinking about the structure and connectedness of concepts.

Chapter 2 The Number System

This chapter develops the number system to help students to see the connections. Intention was especially put into helping readers to see that fractions are just numbers and not some completely different concept. Section 2.1 includes the development of children's understanding of numeration and its historical development, both of which students find fascinating. Exploration 2.2 (Alphabitia) is one of the most powerful explorations we have used. Most students report this to be the most significant learning and turning point in the semester. The exploration unlocks important understandings related to numeration, which the text supports by discussing the evolution of numeration systems and exploring different bases, giving students a strong understanding of the experience of young students trying to learn math. Section 2.2 is dedicated to developing the concept of fractions; Section 2.3 then develops the concepts of decimals, integers, and real numbers.

Chapter 3 Understanding Addition and Subtraction

This chapter connects the operations of addition and subtraction, as well as showing how adding fractions is related to adding whole numbers, which is related to adding decimals, and so forth. Section 3.1 develops models and pictorial representations for the addition of whole numbers, including doing so with other bases to further develop place value concepts. Section 3.2 develops the concept of subtraction of whole numbers in a similar manner. Section 3.3 is dedicated to developing addition and subtraction of fractions; Section 3.4 repeats the experience with decimals and integers.

Students see how the concepts of the operations, coupled with an understanding of base ten, enable them to understand how and why procedures that they have performed by memorization for years actually work. In addition to making sense of standard algorithms, we present alternative algorithms in both the text and explorations. Students have found these algorithms to be both enlightening and fascinating.

Chapter 4 Understanding Multiplication and Division

This chapter connects the operations of multiplication and division with different number sets. It is organized similarly to Chapter 3 in that the first section focuses on multiplication models and understanding with whole numbers and then follows it with division in the second section. Section 4.3 is focused on multiplication and division of fractions, while Section 4.4 does the same with decimals and integers.

Chapter 5 Proportional Reasoning

The investigations and explorations in Chapter 5 are conceptually rich and provide many real-life examples so that students can enjoy developing an understanding of multiplicative relationships. We look at percentages as ratios and also make connections with fractional thinking.

Chapter 6 Algebraic Thinking

Chapter 6 explores patterns, the concept of a variable, and solving equations and inequalities using different models, including Singapore bar models. The four sections are arranged under the National Council of Teachers of Mathematics (NCTM) algebra structure of understanding patterns, relations, and functions; representing and analyzing math situations and structures using algebraic symbols; using mathematical models to represent and understand quantitative relationships; and analyzing change in various contexts.

Chapter 7 Understanding Statistics and Probability

In this chapter, students carefully walk through the stages of defining a question, collecting data, interpreting data, and then presenting data. We are particularly excited that the investigations with the concepts of mean and standard deviation remain successful with students. As a result, students can express these ideas conceptually instead of simply reporting the procedure. The sections in this chapter have been rearranged using the NCTM Standards as a framework. Probability has been put in the same section as counting to help students see how these concepts are related.

Chapter 8 Geometry as Shape

In this chapter, you have the option of introducing geometry through explorations with tangrams, Geoboards, or pentominoes. This more concrete introduction allows students with unpleasant or failing memories of geometry to build confidence and understanding while engaging in rich mathematical explorations.

Chapter 9 Geometry as Measurement

This chapter addresses measurement from a conceptual framework (such as identify the attribute, determine a unit, and determine the amount in terms of a unit) and a historical perspective. Both the explorations and investigations get students to make sense of measurement procedures and to grapple with fundamental measurement ideas. The text looks at the larger notion of measurement, presents the major formulas in a helpful way, and illustrates different problem-solving paths.

Chapter 10 Geometry as Transforming Shapes

The geometric transformations that we explore in Chapter 10 can be some of the most interesting and exciting topics of the course. Quilts and tessellations both spark lots of interest and provoke good mathematical thinking. The text develops concepts and introduces terms that help students to refine understanding that emerges from explorations.

PREFACE

Owning versus Renting

This course is about developing and *retaining* the mathematical knowledge that students will need as beginning mathematics teachers. We prefer to say that we are going to *uncover* the material rather than *cover* the material. The analogy to archaeology is useful. When archaeologists explore a site, they carefully *uncover* the site. As time goes on, they see more and more of the underlying structure. This is exactly what can and should happen in a mathematics course. When this happens, students are more likely to *own* rather than to *rent* the knowledge.

There are three ways in which this textbook supports owning versus renting:

1. Knowledge is constructed.
2. Connections are reinforced.
3. Problems appear in authentic contexts.

1. Constructing Knowledge

When students are given problems, such as appear in Investigations throughout the textbook, that involve them in grappling with important mathematical ideas, they learn those ideas more deeply than if they are simply presented with the concepts via lecture and then are given problems for practice. There is a need to shift the focus from students studying mathematics to students doing mathematics. That is, students are looking for patterns, making and testing predictions, making their own representations of a problem, inventing their own language and notation, and so on. Developing concepts starting with a concrete investigation, moving to a pictorial mode before moving to the abstract helps students to construct their own understanding of the concepts.

Investigation 1.1f (Pigs and Chickens) confronts a common misconception—that there is one right way to solve math problems—by exploring five valid solution paths to the problem. This notion of multiple solution paths is an important part of the book.

INVESTIGATION 1.1f **Pigs and Chickens**

A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, "I count 24 heads and 80 feet. How many pigs and how many chickens are out there?"

Before reading ahead, work on the problem yourself or, better yet, with someone else. Close the book or cover the solution paths while you work on the problem.

Compare your answer to the solution paths below.

DISCUSSION

STRATEGY 1 **Use random trial and error**

One way to solve the problem might look like what you see in Figure 1.4.

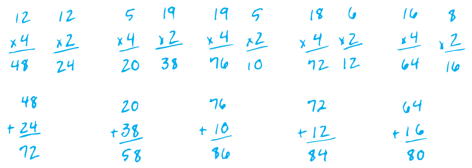


Figure 1.4

2. Reinforcing Connections

Understanding can be defined in terms of connections; that is, the extent to which you *understand* a new idea can be seen by the *quality* and *quantity* of connections between that idea and what you already know. There are two ways in which connections are built into the structure of the text.

1. Mathematical connections

Owning mathematical knowledge involves connecting new ideas to ideas previously learned. It also involves truly understanding mathematics, not just memorizing formulas and definitions.

- **CONNECTIONS AMONG CONCEPTS ARE EMPHASIZED**

Throughout the book, connections between concepts are developed and presented as crucial for deep understanding. Chapter 2 focuses on developing numeracy by showing the connections between the sets of numbers. Chapters 3 and 4 both develop the connections between the operations and continue the connections between the different number sets by showing that adding one set of numbers is a similar concept to adding another set of numbers. Connections are shown between Statistics and Probability in Chapter 7, while the Geometry chapters also continue the numeracy, as well as connect to the graphs covered in Statistics.

- **THE *HOW* IS CONNECTED TO *WHY***

In this way, students know not only how the procedure works but also why it works. For example, students understand why we move over when we multiply the second row in whole number multiplication; they realize that "carrying" and "borrowing" essentially equate to trading tens for ones or ones for tens; they understand why we first find a common denominator when adding fractions; and they see that π is how many times you can wrap any diameter around the circle.

2. Connections to children's thinking

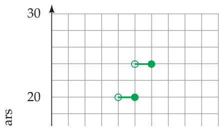
In this book you will see a strong focus on children's thinking, for two reasons. First, much work with teachers focuses on the importance of listening to the students' thinking as an essential part of good teaching. If students experience this in a math course, then by the time they start teaching, it is part of how they view teaching. Second, when students see examples of children's thinking and see connections between problems in this course and problems children solve, both the quality and quantity of the students' cognitive effort increase.

If it is true that we tend to teach the way we were taught, we need to consider the experience we are creating in these courses.

3. Authentic Problems

Although most texts have many "real-life" problems, this text differs in how those problems are made and presented.

In Section 6.3, the question of paying a baby-sitter is explored. This situation is often portrayed as a linear function: for example, if the rate is \$10 per hour, $y = 10x$. However, in actuality, it is not a linear function but rather a stepwise function.



A closer look at paying the baby-sitter At first glance, the question of how much to pay the baby-sitter is simple: Multiply the hours sat by 8. However, let us use the problem-solving strategy "act it out" to examine this problem more closely. For example, what if Ellen baby-sat from 7 to 11:15? How much would you pay her? Think before reading on. . . .

Some people say \$32. Some people say \$36—they round up to the nearest half-hour. In actuality, different people have different ways of determining how much to pay a baby-sitter. Let us examine the case of a couple, who rounds up the time to the nearest half-hour. We could now represent their process for paying the baby-sitter in each of the ways we have just examined. Is the relationship between time sat and dollars earned a functional relationship in the case of the couple? Think and then read on. . . .

If you were to graph this relationship, what would the graph look like? Try to make your own graph before reading on. . . .

Take a look at the graph on the left and see whether you can make sense of it before reading on.

If you are having trouble making sense of the graph, consider a few examples.

Similarly, when determining the cost of carpeting a room, the solution path is often presented as dividing the area of the room by the cost per square yard; again, this is not how the cost is actually determined.

In this book, you will find many problems—problems we have needed to solve, problems friends have had, problems children have had, problems we have read about—where the content fits with the content of this course.

You will find problems in the text where students are asked to state the assumptions they make in order to solve the problem (such as Section 5.2, Exercises 33 to 38). You also will find problems that have the messiness of "real-life" problems, where the problem statement is ambiguous, too little or too much information is given, or the information provided is contradictory.

Problems 33–38 require you to make some assumptions in order to determine an answer. Describe and justify the assumptions you make in determining your answer.

33. Let's say that you read in the newspaper that last year's rate of inflation was 7.2%.

  a. If your grocery bill averaged \$325 per month last year, about how much would you expect your grocery bill to be this year?

  b. Let's say you received a \$1200 raise, from \$23,400 per year to \$24,600 per year. Did your raise keep you ahead of the game, or are you falling behind?

34. A government agency is recommending that the legal definition of drunk driving be reduced from an alcohol blood content of 0.08 to 0.05. Explain why some might consider this a little drop and others might consider it a big drop. What do you think?

35. Which would you prefer to see on a sale sign at a store: \$10 off or 10% off? Explain your choice.

36. Refer to Investigation 5.2b. Jane still doesn't understand the problem. Roberto tries to help her make sense of the problem by saying that the 8% means that if we were to select 100 students at the college, 8 of them would be working full-time. What do you think?

Features

What do you think?

What-do-you-think questions appear at the start of each section to help students focus on key ideas or concepts that appear within the sections.

6.1 Understanding Patterns, Relations, and Functions

What do you think?

- How are patterns related to algebraic thinking?
- What are some examples of functions in everyday life?
- What is a reason for developing algebraic thinking in elementary school?

Investigations

Investigations are the primary means of instruction, uniquely designed to promote active thinking, reasoning, and construction of knowledge. Each investigation presents a problem statement or scenario that students work through, often to uncover a mathematical principle relevant to the content of the section. The "Discussion" that follows the problem statement provides a framework for insightful solution logic.

INVESTIGATION 3.1h Children's Mistakes

The problem below illustrates a common mistake made by many children as they learn to add. Understanding how a child might make that mistake and then going back to look at what lack of knowledge of place value, of the operation, or of properties of that operation contributed to this mistake is useful. What error on the part of the child might have resulted in this wrong answer?

The problem: $38 + 4 = 78$

DISCUSSION

In this case, it is likely that the child lined up the numbers incorrectly:

$$\begin{array}{r} 4 \\ +\ 38 \\ \hline 78 \end{array}$$

Giving other problems where the addends do not all have the same number of places will almost surely result in the wrong answer. For example, given $45 + 3$, this child would likely get the answer 75. Given $234 + 42$, the child would likely get 654. In this case, the child has not "owned" the notion of place value. Probably, part of the difficulty is not knowing expanded form (for example, that 38 means $30 + 8$—that is, 3 tens and 8 ones). An important concept here is that we need to add ones to ones, tens to tens, and so on. Base ten blocks provide an excellent visual for this concept as students can literally see why they cannot add 4 ones to 3 tens.

Questions in the Text

To encourage active learning outside of the Investigations, questions appear embedded within the text, often accompanied by the icon. These "thinking" questions require students to pause in their reading to reflect or to complete a short exercise before continuing. Answers to these questions can be found in Appendix B (available in MindTap).

Translate the following Babylonian numerals into our system. Check your answers in Appendix B.

1. 

2. 

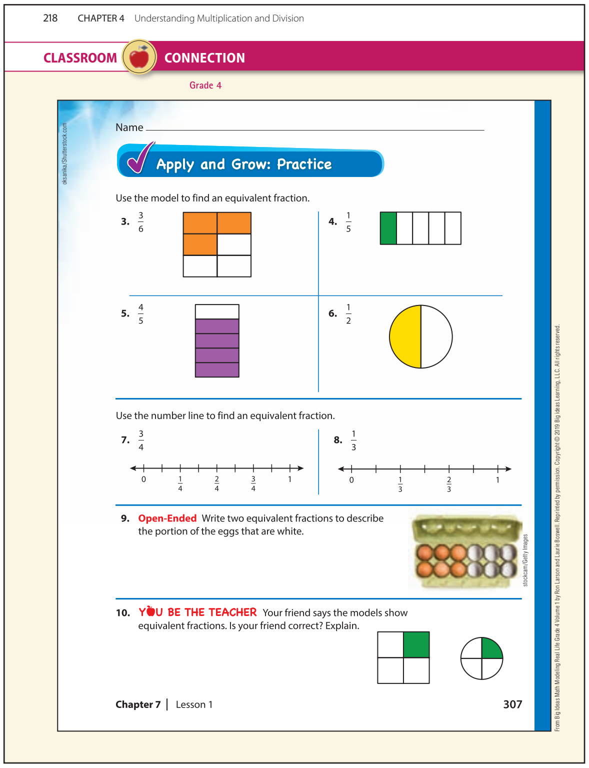
3. 

Translate the following amounts into Babylonian numerals.

4. 1202 **5.** 304

Classroom Connections

Assignments from actual elementary/middle school books appear throughout so that students can see how the material they are learning will directly apply. Connections are also found in the exercises that highlight children's work.


218 CHAPTER 4 Understanding Multiplication and Division

CLASSROOM CONNECTION

Grade 4

Name

Apply and Grow: Practice

Use the model to find an equivalent fraction.

3. $\frac{3}{6}$ 4. $\frac{1}{5}$

5. $\frac{4}{5}$ 6. $\frac{1}{2}$

Use the number line to find an equivalent fraction.

7. $\frac{3}{4}$ 8. $\frac{1}{3}$

9. Open-Ended Write two equivalent fractions to describe the portion of the eggs that are white.

10. YOU BE THE TEACHER Your friend says the models show equivalent fractions. Is your friend correct? Explain.

Chapter 7 | Lesson 1 307

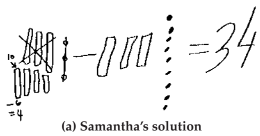
oksanika/Shutterstock.com

stockcam/Getty Images

From Big Ideas Math Modeling Real Life Grade 4 Volume 1 by Ron Larson and Laurie Boswell. Reprinted by permission. Copyright © 2019 Big Ideas Learning, LLC. All rights reserved.

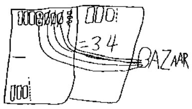
11. Following are two children's solutions to 73 − 39. Study each child's work, and then describe how that child would find the answer to 65 − 28. Then explain why that method works.

a.



(a) Samantha's solution

b.



(b) Alice's solution

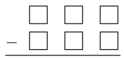
Source: *Teaching Children Mathematics*, by the National Council of Teachers of Mathematics December, 2001, p. 231.

Section Exercises

The exercises are designed to give students a deeper sense and awareness of the kinds of problems their future students are expected to solve at various grade levels, as well as to increase their own proficiency with the content. *From Standardized Assessments* exercises derive from exams such as the NECAP and NAEP to give students a sense of the types of questions found on diverse national exams at various grade levels. Questions are also included from the Smarter Balanced Assessment Consortium which is developing assessments for Common Core State Standards.

Source: National Council of Teachers of Mathematics.

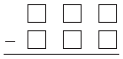
28. Place the digits 1, 2, 3, 6, 7, and 8 in the boxes to obtain



a. The greatest difference

b. The least difference

29. Choose among the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 to make the difference 234. You can use each digit only once. How many different ways can you make 234?



30. With three boys on a large scale, it read 170 pounds. When Adam stepped off, the scale read 115 pounds. When both Adam and Ben stepped off, the scale read 65 pounds. What is the weight of each boy?

31. A mule and a horse were carrying some bales of cloth. The mule said to the horse, "If you give me one of your bales, I shall carry as many as you." "If you give me one of yours," replied the horse, "I will be carrying twice as many as you." How many bales was each animal carrying?

FROM STANDARDIZED ASSESSMENTS

NECAP 2006, Grade 5

36. Mrs. Lombardi had 2 hours to prepare for a party. The chart below shows the amount of time she spent completing different tasks.

TIME MRS. LOMBARDI SPENT ON DIFFERENT TASKS

Task	Time
Decorated cake	20 minutes
Made punch	15 minutes
Made sandwiches	50 minutes
Put up balloons	?

How much time did Mrs. Lombardi have to put up the balloons? (1 hour = 60 minutes)

a. 15 minutes **b.** 25 minutes

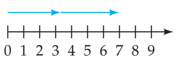
c. 35 minutes **d.** 45 minutes

Exercises 3.1

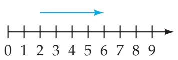
1. Reread the 8 Mathematical Practices of the CCSS in Section 1.1, or on the corestandards.org website and write about relationships between them and what we did in this section.

2. For each number line problem below, identify the computation it models and briefly justify your answer.

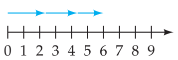
a.



b.



c.



3. a. Draw a picture that shows that addition is commutative.

b. Draw a picture that shows that addition is associative.

9. Through illustrations, demonstrate how to solve these problems with manipulatives.

a. $\begin{array}{r} 76 \\ +47 \\ \hline \end{array}$ **b.** $\begin{array}{r} 524 \\ +268 \\ \hline \end{array}$

10. Below is an addition algorithm from an old text. Explain why it works.

$$\begin{array}{r} 36 \\ +48 \\ \hline 14 \\ 7 \\ \hline 84 \end{array}$$

11. How does this addition algorithm work? Imagine someone asking, "How do you know where to put the 1 and the 3 in the 13?"

Section Summary

Each section ends with a summary that reviews the main ideas and important concepts discussed.

SUMMARY 3.2

We have now examined addition and subtraction rather carefully. In what ways do you see similarities between the two operations? In what ways do you see differences? Think and then read on. . . .

One way in which the two processes are alike is illustrated with the part–whole diagram used to describe each operation. These representations help us to see connections between addition and subtraction. In one sense, addition consists of adding two parts to make a whole. In one sense, subtraction consists of having a whole and a part and needing to find the value of the other part.

We see another similarity between the two operations when we watch children develop methods for subtraction; it involves the "missing addend" concept. That is, the problem $c - a$ can be seen as $a + ? = c$.

We saw a related similarity in children's strategies. Just as some children add large numbers by "adding up," some children subtract larger numbers by "subtracting down."

Earlier in this section, subtraction was formally defined as $c - b = a$ if $a + b = c$. The negative numbers strategy that some children invent brings us to another way of defining subtraction, which we will examine further in Section 3.2 when we examine negative numbers. That is, we can define subtraction as adding the inverse: $a - b = a + -b$.

A very important way in which the two operations are different is that the commutative and associative properties hold for addition but not for subtraction.

Looking Back

Each chapter concludes with *Looking Back*—a study tool that brings together all the important points from the chapter. *Looking Back* includes *Questions to Summarize Big Ideas*, which ask students to reflect on the main ideas from the chapter; *Chapter Summary*, which lists major take-aways and terminology from the chapter; and *Review Exercises*, which provide an opportunity for students to put concepts from the chapter into practice.

LOOKING BACK on Chapter 3

QUESTIONS TO SUMMARIZE BIG IDEAS

1. What are some of the different models for addition and subtraction?
2. How can you use base ten blocks to model the algorithms for addition and subtraction of whole numbers and decimals?
3. How are these models similar and different in a base other than ten?
4. Which algorithms for the operations are different from what you learned in elementary school?
5. Look back at the Mathematical Practices of the Common Core State Standards. In what ways did you engage in those practices during this c
6. What parts of this chapter are less clea

CHAPTER 3 SUMMARY

1. There are many concrete and pictorial models for addition and subtraction.
2. Each operation has multiple meanings.
3. Many algorithms have been developed to enable us to compute more efficiently.
4. The standard algorithm for each operation does not connect equally well to each meaning of the operation.
5. Being able to make sense of algorithms requires:
   - The ability to apply base ten and place value concepts
   - The ability to compose and decompose the numbers (for example, to use expanded form)
6. Patterns enable us to understand the operations more deeply.
7. In many real-life problems, the answer depends on knowing how to interpret one's computation.
8. Being able to perform mental math and to estimate requires:
   - The ability to apply base ten and place value concepts
   - The ability to compose and decompose the numbers (for example, to use expanded form)
   - The ability to apply properties of the operations, especially the commutative, associative, and distributive properties
9. In real-life problem solving, one needs to know when to find an exact answer and when to find an estimate.

Addition properties:
identity 106 | commutative 107
associative 107 | closure 107

Algorithms for addition:
common 116 | lattice 117

Other terminology:
matrix 107 | composed 117
algorithm 114 | decompose 117

Section 3.2: Understanding Subtraction of Whole Numbers

Subtraction contexts:
take-away 128 | comparison 129
missing addend 129

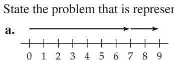
Subtraction terminology:
subtraction 130 | minuend 130
subtrahend 130 | difference 130

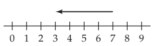
Subtraction models:
number line 132

Section 3.3: Understanding Addition and Subtraction with Fractions

REVIEW EXERCISES Chapter 3

1. State the problem that is represented in each case below:

a.


b.


Explain why it works, as though to a parent who only the traditional right-to-left algorithm.

832
+ 549
1300
70

Explorations

The icon that appears throughout the text references additional activities that may be found in the *Explorations Manual*. Explorations present new ideas and concepts for students to engage with "hands on." The *Explorations Manual* is now available for free use on both the Instructors and Students website for the book. It can be accessed at login.cengage.com.

Explorations Manual
1.5

MP 4: Model with Mathematics.

- Mathematically proficient students can solve real-life problems, which in elementary school includes being able to write a multiplication equation to solve a problem.
- They are able to identify important information in a real-life problem and analyze relationships using tools.
- They can use models to draw conclusions, make predictions, and reflect on and adjust the effectiveness of the model.

MP 5: Use Appropriate Tools Strategically.

- Mathematically proficient students can use a variety of tools such as concrete models and technology to and solutions.
- They are able to make good decisions about when to use each of these tools and how to effectively use them.
- They are able to use a variety of tools to investigate and develop their understanding of ideas.

SUPPLEMENTS	
FOR THE STUDENT	**FOR THE INSTRUCTOR**
	Instructor's Edition **(ISBN: 978-0-357-02551-2)** The Instructor's Edition includes answers to all exercises in the text, including those not found in the student edition. (Print)
Student Solutions Manual **(ISBN: 978-0-357-02553-6)** Go beyond the answers—see what it takes to get there and improve your grade! This manual provides worked-out, step-by-step solutions to the odd-numbered problems in the text. This gives you the information you need to truly understand how these problems are solved.	**Complete Solutions Manual** The Complete Solutions Manual provides worked-out solutions to all of the problems in the text. In addition, instructors will find helpful aids such as "Teaching the Course," which shows how to teach in a constructivist manner. "Chapter by Chapter Notes" provide commentary for the *Explorations* manual as well as solutions to exercises that appear in the supplement. This manual can be found on the Instructor Companion Site.
***Explorations, Mathematics for Elementary School Teachers,* 7e** **(ISBN: 978-0-357-02552-9)** This manual contains open-ended activities for you to practice and apply the knowledge you learn from the main text. When you begin teaching, you can use the activities as models in your own classrooms. Posted on the Student Resource Center.	***Explorations, Mathematics for Elementary School Teachers,* 7e** **(ISBN: 978-0-357-02552-9)** This manual contains open-ended activities for students to practice and apply the knowledge they learn from the main text. When students begin teaching, they can use the activities as models in their own classrooms. Posted on the Instructor Companion Site at login.cengage.com
Enhanced WebAssign® Instant Access Code: 978-0-357-04379-0 Printed Access Card: 978-0-357-04377-6 Enhanced WebAssign combines exceptional mathematics content with the powerful online homework solutions, WebAssign. Enhanced WebAssign engages students with immediate feedback, rich tutorial content, and an interactive, fully customizable eBook, the Cengage MindTap Reader, which helps students to develop a deeper conceptual understanding of their subject matter.	**Enhanced WebAssign®** Instant Access Code: 978-0-357-04379-0 Printed Access Card: 978-0-357-04377-6 Enhanced WebAssign combines exceptional mathematics content with the powerful online homework solutions, WebAssign. Enhanced WebAssign engages students with immediate feedback, rich tutorial content, and an interactive, fully customizable eBook, the Cengage MindTap Reader, which helps students to develop a deeper conceptual understanding of their subject matter. See www.cengage.com/ewa to learn more. Virtual manipulatives and more conceptual questions have been added and will continue to be updated annually.
	Instructor Companion Site Everything you need for your course is in one place! This collection of book-specific lecture and classroom tools is available online via www.cengage.com/login. Access and download Lecture Notes PowerPoints, images, Complete Solutions Manual, and more.
	Cengage Learning Testing Powered by Cognero® Cognero is a flexible, online system that allows you to author, edit, and manage test bank content; create multiple test versions in an instant; and deliver tests from your LMS, your classroom, or wherever you want. This is available *online* via www.cengage.com/login.

Acknowledgments

We would like to thank the reviewers of this edition:

Robin Ayers, Western Kentucky University; Mary Beard, Kapiolani Community College; Ivette Chuca, El Paso Community College–Valle Verde; Kim Johnson, West Chester University; Mark Kuhlman, Casper College; Senfeng Liang, University of Wisconsin–Stevens Point; Judith Macks, Towson University; Ann-Marie Spinelli, Central Connecticut State University; Allison Sutton, Austin Community College; and Carol Williams, Texas Tech University.

In addition, we would like to thank the many reviewers of previous editions noted below for their thoughtful and helpful comments throughout development: Marilyn Ahrens, Missouri Valley College; Timothy Comar, Benedictine University; Edward DePeau, Central Connecticut State University; Sue Ann Jones Dobbyn, Pellissippi State Community College; April Hoffmeister, University of Illinois; Judy Kasabian, El Camino College; Cathy Liebars, The College of New Jersey; Kathleen McDaniel, Buena Vista University; Ann McCoy, University of Central Missouri; Martha Meadows, Hood College; Anthony Rickard, University of Alaska, Fairbanks; Mark Schwartz, Southern Maine Community College; Sonya Sherrod, Texas Tech University; Allison Sutton, Austin Community College; Osama Taani, Plymouth State University; Michael Wismer, Millersville University; Ronald Yates, College of Southern Nevada; Andrew T. Wilson, Austin Peay State University; Anita Goldner, Framingham State College; Art Daniel, Macomb Community College; Bernadette Antkoviak, Harrisburg Area Community College; Beverly Witman, Lorain County Community College; Charles Dietz, College of Southern Maryland; Clare Wagner, University of South Dakota; Deborah Narang, University of Alaska–Anchorage; Dennis Raetzke, Rochester College; Donald A. Buckeye, Eastern Michigan University; Doug Cashing, St. Bonaventure University; Dr. Connie S. Schrock, Emporia State University; Elise Grabner, Slippery Rock University; Elizabeth Cox, Washtenaw Community College; Forrest Coltharp, Pittsburg State University; Fred Ettline, College of Charleston; Gary Goodaker, West Community and Technical College; Gary Van Velsir, Anne Arundel Community College; Glenn Prigge, University of North Dakota; Helen Salzberg, Rhode Island College; Isa S. Jubran, SUNY College at Cortland; J. Normon Wells, Georgia State University; J.B. Harkin, SUNY College at Brockport; James E. Riley, Western Michigan University; Jane Ann McLaughlin, Trenton State College; Jean M. Shaw, University of Mississippi; Jean Simutis, California State University–Hayward; Jeanine Vigerust, New Mexico State University; Jerry Dwyer, University of Tennessee–Knoxville; Jim Brandt, Southern Utah University; John Long, University of Rhode Island; Juan Molina, Austin Community College; Julie J. Belock, Salem State College; Karen Gaines, St. Louis Community College; Karla Karstens, University of Vermont; Kathy C. Nickell, College of DuPage; Larry Feldman, Indiana University of Pennsylvania; Larry Sowder, San Diego State University; Lauri Semarne; Lawrence L. Krajewski, Viterbo College; Lew Romagnano, Metropolitan State College of Denver; Linda Beller, Brevard Community College; Linda Herndon, Benedictine College; Lois Linnan, Clarion University; Lorel Preston, Westminster College; Loren P. Johnson, University of California–Santa Barbara; Lynette King, Gadsden State Community College; Marvin S. Weingarden, Madonna University; Mary Ann Byrne Lee, Mankato State University; Mary J. DeYoung, Hope College; Mary Lou Witherspoon, Austin Peay State University; Mary T. Williams, Francis Marion University; Mary Teagarden, Mesa College; Matt Seeley, Salish Kootenai College; Maureen Dion, San Joaquin Delta Community College; Merle Friel, Humboldt State University; Merriline Smith, California State Polytechnic University; Michael Bowling, Stephens College; Nadine S. Bezuk, San Diego State University; Peter Berney, Yavapai College; Peter Incardone, New Jersey City University; Robert F. Cunningham, Trenton State College; Robert Hanson, Towson State University; Rebecca Wong, West Valley College; Ronald Edwards, Westfield State University; Ronald J. Milne, Gashen College;

Sandra Powers, College of Charleston; Stephen P. Smith, Northern Michigan University; Stuart Moskowitz, Humboldt State University; Susan K. Herring, Sonoma State University; Tad Watanabe, Towson State University; Tess Jackson, Winthrop University; Vena Long, University of Missouri at Kansas City; and William Haigh, Northern State University.

We offer deep thanks to the people at Cengage and beyond who offered guidance, support, and expertise in ensuring the quality of the resulting product.

These individuals include Spencer Arritt, Adrian Daniel, and Mona Zeftel.

We would also like to extend our gratitude to Mary Beard and Neil Starr who served in an advisory capacity on this edition, offering valuable suggestions and feedback.

We would both like to thank our families who have been there for support throughout the long hours of working on this book.

We also send a special thank you to the thousands of students we have taught over the years and all that you have taught us. Thank you!

1 Foundations for Learning Mathematics

Knowing mathematics means being able to use it in purposeful ways. To learn mathematics, students must be engaged in exploring, conjecturing, and thinking rather than only in rote learning of rules and procedures. Mathematics learning is not a spectator sport. When students construct personal knowledge derived from meaningful experiences, they are much more likely to retain and use what they have learned. This fact underlies [the] teacher's new role in providing experiences that help students make sense of mathematics, to view and use it as a tool for reasoning and problem solving.[1]

—National Council of Teachers of Mathematics

SECTION 1.1 What Is Mathematics?

What do you think?

- Respond to the prompt: Mathematics is ________________.
- Describe a few of your experiences learning mathematics.
- What does it mean to learn mathematics?

You are at the beginning of a course where you will re-examine elementary school mathematics to understand these concepts on a much deeper level, and to learn why the mathematical procedures and formulas actually work. On this journey, you will learn several ways to see and think about concepts and procedures that you may have previously simply memorized. This deeper understanding will lead to increased confidence and comfort level with mathematics. Mathematics is both beauty and truth. Two plus two always equals four. The distance around a circle is always a little more than three (actually pi) times the distance across the circle. Throughout this book, we hope you will appreciate more and more of the beautiful truths of mathematics.

Your approach to learning and teaching mathematics depends on the attitudes and beliefs you bring; in subtle and not so subtle ways, they may affect your learning of math and you may pass these beliefs along when you enter the classroom as a teacher. Reflect on how you answered the questions above. Whatever your feelings about mathematics, consider where these feelings come from. Research suggests that people who have math anxiety can relate it back to a teacher and/or experience in their elementary or middle school years. Think about the best math teacher you have had as well as the worst math teacher you have had. Consider the skills and qualities that each of them had that led to your experience of them. What skills and qualities do you have and need to further develop to become an excellent math learner and teacher?

Mathematical Growth Mindset

Do you have a fixed mindset or a growth mindset when it comes to learning math? Take this survey to find out.

Seven pairs of statements are given in Table 1.1. Score your beliefs in the following manner:

- If you strongly agree with the statement in column A, record a 1.
- If you agree with the statement in column A more than column B, record a 2.
- If you agree with the statement in column B more than column A, record a 3.
- If you strongly agree with the statement in column B, record a 4.

Table 1.1

Column A		Column B
1. There will be many problems in this book that I won't be able to solve, even if I try really hard.	1 2 3 4	**1.** I believe that if I try really hard, I can solve virtually every problem in this book.
2. There is only one way to solve most problems.	1 2 3 4	**2.** There is usually more than one way to solve most problems.
3. The best way to learn is to memorize the different kinds of problems and the steps to solve them.	1 2 3 4	**3.** The best way to learn is to make sure that I understand the concepts and each step I take to solve the question.
4. Some people have mathematical minds and some don't. Nothing they do can *really* make a difference.	1 2 3 4	**4.** Everyone can learn math with the right opportunities to learn and hard work.
5. I get frustrated when I make a mistake and want to give up.	1 2 3 4	**5.** I am comfortable making mistakes because mistakes help me to learn.
6. Mathematics is about getting the right answer by quickly recalling math facts.	1 2 3 4	**6.** Mathematics is about problem solving and critical thinking.
7. I only plan to teach very young children, so all I need to know is basic numbers and operations.	1 2 3 4	**7.** I need to develop a deeper understanding of all elementary school math topics.
	Total_______	

In Table 1.1, the statements in column A indicate a fixed mindset, and the statements in column B indicate the corresponding growth mindset. If you take the arithmetic average, or *mean*, of your scores (by adding up your scores and dividing by seven), you will get a number that we could call your belief index. If your belief index is low, you are currently in more of a fixed mindset.

A main goal of this book is to help you understand math in a conceptual and flexible way to help you foster a "growth mindset" toward math and increase your self-confidence about math, which in turn you can pass on to your students. Many people see math as a set of rules and formulas to memorize instead of connected concepts that make sense. Deepening your understanding of mathematics through your studies will help you to have a positive attitude about math. The essence of a math growth mindset is to approach math conceptually, seek understanding, and believe that everyone can learn math.

If you have been told that you do not have a math brain, or learned to believe this—it is not true. Recent brain research shows us that, with the right learning opportunities and beliefs, everyone can learn math and reach high levels of math. Henry Ford is quoted as saying, "Whether you believe you can do a thing or not, you are right." As you engage with this book, believe you can make sense of the math concepts. If you catch yourself saying anything like "I am not good at math," change that to "I can learn math." By changing your words, you can change your mindset. That is the first step in learning and creating a growth mindset. Approach the concepts in this book with an open mind toward learning and a belief that you can learn the concepts. Your future students are depending on you to deepen your understanding of math and develop a growth mindset so that you can teach them.

In our current performance model of schools, where getting the right answer is valued, mistakes are seen as bad. However, recent brain research provides evidence that mistakes actually help our brain to grow. Our brains grow even more by making a mistake and learning from it (Moser, et al., 2011). When you follow a path on a math problem that does not get you to the answer, go back, see where you went down a nonproductive path, and correct the way you were thinking about it. Studies of successful people show that they make many more mistakes than unsuccessful people. Therefore, as you read through this book, be comfortable exploring ideas and making mistakes, learn from your mistakes, persevere through the solution, and success will be yours.

Owning Versus Renting

You will notice that I often ask you to stop, think, and write some notes. I really mean it! Many students rent what they have learned just long enough to pass the test. However, within days or weeks of the final exam, it's gone. If you make sense of what you are learning and make connections, then you will remember the concepts. One of the important differences between students who own what they learn and those who simply rent is that those who own the knowledge tend to be active readers.

Mathematical Knowledge for Teaching

Having a mathematical growth mindset is especially important for teachers, both in their own learning and their ability to help their students.

INVESTIGATION 1.1a More than One Way to Multiply?

First, multiply 49×25 using any method you choose. Then consider how the following students solved the problem.

What method do you think each student is using in each solution below? Do you think each method will work for any two whole numbers?

Note: Whenever you see the pencil icon in this book, stop and think and briefly write your thoughts before reading on. Students who take the time to think and write after these points (or at least to pause and think) say that it makes a big difference in how much they learn.

Student A:

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 245 \\ 98\ \ \\ \hline 1225 \end{array}$$

Student B:

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 45 \\ 200 \\ 180 \\ 800 \\ \hline 1225 \end{array}$$

DISCUSSION

We will look much more deeply into multiplication later in this book. We use this example to illustrate how part of the teacher's role is to first have a math mindset that there are different methods for solving problems; to be able to develop, understand, and communicate different strategies; and to see the connections between them. This book is not about methods of teaching; it is about developing a deep and flexible understanding of math and developing a mathematical growth mindset.

Student A used the method of 5×49 and 20×49 that many of us learned in elementary school, especially if we went to school in the United States. Different and equally effective methods are taught in other countries, and we will look at some of these in this text to deepen our understanding. Some of us learned to write the second row in the multiplication as 980, and some were taught to move over a space as is shown here. Why do we do this? Because we are multiplying 49×20, not 49×2, so we write it as either 980 or as 98 with a space in the ones place. Either way, the 8 is in the tens place and the 9 is in the hundreds place when we add to get our final answer. Place value is a key concept to deeply understand in mathematics.

We can use the properties and split the numbers up into their parts to figure this out.

$$49 \times 25 = 49 \times (20 + 5) = 49 \times 20 + 49 \times 5 = 980 + 245 = 245 + 980 = 1225$$

Student B used a method that is sometimes called partial products of $9 \times 5 = 45$, $40 \times 5 = 200$, $9 \times 20 = 180$, and $40 \times 20 = 800$. Another way to write this, which uses the distributive property more clearly, is:

$$49 \times 25 = (40 + 9) \times (20 + 5) = 40 \times 20 + 40 \times 5 + 9 \times 20 + 9 \times 5 = 600 + 200 + 180 + 45 = 1225$$

The methods of both students are valid. One of the major ideas in this text is that there are multiple ways (often called "solution paths") to get to an answer. There is no ONE "right" way to solve any math problem. This may be different from what you have always thought about mathematics. You may have even had teachers who marked you "wrong" if you did not solve it their way.

As a teacher of elementary school mathematics, a deep understanding of mathematics will enable you to respond to the above type of situation in an elementary school classroom. Simply being able to get the right answer is not sufficient. Teachers need a specialized understanding of mathematics that is flexible, connected, and conceptual.

Teachers use mathematics every day in the classroom, but in different ways than other people. In 1986, Lee Shulman used the term "pedagogical content knowledge" to refer to this specialized understanding of mathematics, which includes an understanding of multiple representations and examples, plus an understanding of what ideas may be more difficult for students and why these ideas are more difficult. In 2008, Deborah Loewenberg Ball, Mark Thames, and Geoffrey Phelps developed the framework depicted in Figure 1.1 showing different types of knowledge. This book is focused on the specialized content knowledge, which includes understanding multiple representations and multiple student procedures, and analyzing student errors. It is the type of math content knowledge that teachers draw on every day.

Explorations Manual 1.2

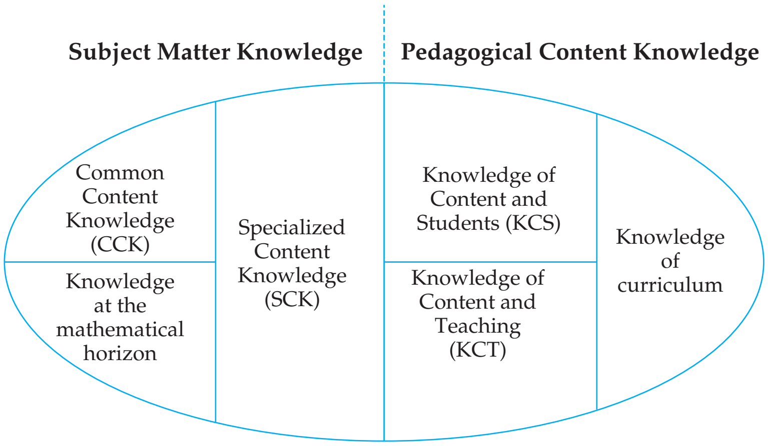


Figure 1.1

Source: Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372–400.

INVESTIGATION 1.1b Understanding Students' Errors

How do you think the answer was produced? What do you think the student was thinking that led to this error?

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 245 \\ 98 \\ \hline 343 \end{array}$$

How does this type of scenario draw upon specialized content knowledge?

DISCUSSION

Analyzing student errors draws upon a specialized understanding of mathematics, in that the teacher needs to understand the mathematics deeply in order to identify the error and then to help the student to correct the misunderstanding.

This student does not understand that in the second step, we are actually multiplying 20 times 49, so it should be 980, not 98.

While most people could get a correct answer for 49×25, teachers need to understand the concept much more deeply. Many of us experienced elementary school mathematics as a series of procedures to memorize (like multiplying 49×25). In order to teach true understanding of mathematics, teachers must develop this specialized content knowledge and a mathematical growth mindset.

Mathematical Problem Solving

A useful metaphor for problem solving is a **toolbox**. Some useful tools to explore in solving math problems include drawing a picture, making a table, using algebra, guess-check-revise, using concrete materials, looking for patterns, solving a simpler problem, and visualization tools. As you go through this course, you will learn new tools and how to more skillfully use tools you already have.

INVESTIGATION 1.1c Real-Life Problem Solving

Consider a few problems you have had in your life, and not necessarily math problems.

What steps did you take to solve these problems?

Use this recollection to make a list of general steps that you take to solve a problem. Then read on to learn about a mathematician that made a similar list. Your list will likely be pretty similar to his.

DISCUSSION

There are many kinds of real-life problems that you may have considered here. One that many of us have dealt with is what kind of car to buy. The first step would be to understand what you need. How many seats do you need? What is your budget? Is gas mileage a priority? What models do you like? This part might be called something like "understand the problem." Did you have a step like this in your list? The second step would be to develop a plan. Where will you look? What research will you do? The third step might be to carry out the plan. Research the best models, look around for the best deals, test drive some models, and find the one you like. The next step, hopefully before actually buying the one you have your heart set on, might be to look back and reflect on whether it really meets your needs, fits your budget, and so on. Read on to see how this process is the same for solving a math problem.

Polya's Four Steps

George Polya developed a framework that breaks down problem solving into four distinguishable steps. In 1945, he outlined these steps in a now-classic book called *How to Solve It*.

Four Steps for Solving Problems

Understanding the Problem

Questions that can be useful to ask:

1. Do you understand what the problem is asking?
2. Can you state the problem in your own words—that is, paraphrase the problem?
3. Have you used all the given information?
4. Can you solve a part of the problem?

Actions that can be helpful:

1. Reread the problem carefully. (Often it helps to reread a problem a few times.)
2. Try to use the given information to deduce more information.
3. Plug in some numbers to make the problem more concrete.

Devising a Plan

Several common strategies:

1. Represent the problem with a diagram (carefully drawn and labeled). Check to see if you used the (relevant) given information. Does the diagram "fit" the problem?
2. Guess–check–revise. Keep track of "guesses" with a table.
3. Make an estimate. The estimate often serves as a useful "check." A solution plan often comes from the estimation process.
4. Make a table (sometimes the clue comes from adding a new column).
5. Look for patterns—in the problem or in your guesses.
6. Be systematic.
7. Look to see if the problem is similar to one you have already solved.
8. If the problem has "ugly" numbers, you may "see" the problem better by substituting "nice" numbers and then thinking about the problem.
9. Break the problem down into a sequence of simpler "bite-size" problems.
10. Act it out.

Carrying Out the Plan

1. Are you keeping the problem meaningful? On each step, ask what the numbers mean. Label your work.
2. Are you bogged down? Do you need to try another strategy?

Looking Back

1. Does your answer make sense? Is the answer reasonable? Is the answer close to your estimate, if you made one?
2. Does your answer work when you check it with the given information? (Note that checking the procedure checks the computation but not the solution.)
3. Can you use a different method to solve the problem?

Using Polya's Four Steps

I encourage you to use Polya's four steps in all of the following ways:

1. Use them as a guide when you get stuck.
2. Don't rent them, buy them. Buying them involves paraphrasing my language and adding new strategies that you and your classmates discover. For example, many of my students have added a step to help reduce anxiety: First take a deep breath and remind yourself to slow down!
3. After you have solved a problem, stop and reflect on the tools you used. Over time, you should find that you are using the tools more skillfully.

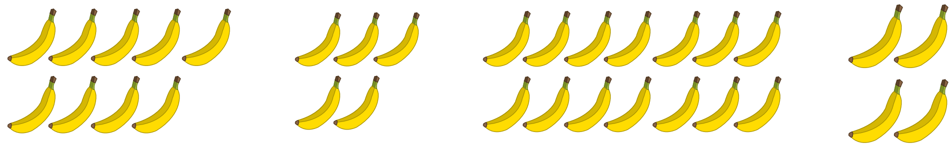
Why Emphasize Problem Solving?

Although Polya described his problem-solving strategies back in 1945, it was quite some time before they had a significant impact on the way mathematics was taught.

CLASSROOM CONNECTION

Grade 2

3. **Structure** There are 9 bananas in a bunch. A monkey eats some. There are 5 bananas left. Which picture shows how many bananas the monkey ate?



4. **Modeling Real Life** You make 4 snow angels and Newton makes 5. Descartes makes 3 more snow angels. How many snow angels are there in all?

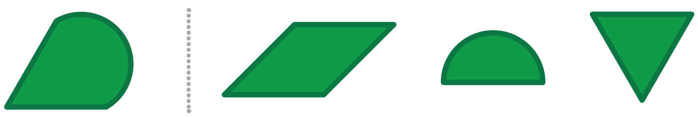
_____ snow angels

5. **Modeling Real Life** Newton has 9 glitter pens. Descartes has 2 fewer than Newton. How many glitter pens do they have in all?

_____ glitter pens

Review & Refresh

6. Which two shapes combine to make the shape on the left?



One of the reasons is that until recently, "problems" were generally defined too narrowly. Many of you learned how to do different kinds of problems—mixture problems, distance problems, percent problems, age problems, coin problems—but never realized that they have many similarities and connections. There has been too great a focus on single-step problems and routine problems instead of on developing a growth mindset in math. Consider the examples from the National Assessment of Educational Progress shown in Table 1.2.

Table 1.2

Problem	Percent correct Grade 11
1. Here are the ages of six children: 13, 10, 8, 5, 3, 3. What is the average age of these children?	72
2. Edith has an average (mean) score of 80 on five tests. What score does she need on the next test to raise her average to 81?	24

Source: Mary M. Lindquist, ed., Results from the *Fourth Mathematics Assessment of the National Assessment of Educational Progress* (Reston, VA: NCTM, 1989), pp. 30, 32.

To solve the first problem, one only has to remember the procedure for finding an average and then use it:

$$\frac{13+10+8+5+3+3}{6}$$

However, there is no simple formula for solving the second problem. Try to solve it on your own and then read on. . . .

To solve this one, you have to have a better understanding of what an average means. One approach is to see that if her average for 5 tests is 80, then her total score for the 5 tests is 400. If her average for the 6 tests is to be 81, then her total score for the 6 tests must be 486 (that is, 81×6). Because she had a total of 400 points after 5 tests and she needs a total of 486 points after 6 tests, she needs to get an 86 on the sixth test to raise her overall average to 81.

Many students consider the second question to be a "trick" question unless the teacher has explicitly taught them how to solve that kind of problem. However, many employers note that problems that occur in work situations are rarely just like the ones in the book. What employers need is more people who can solve the "trick" problems, because, as some may say, "Life is a trick problem!" Let's work through the next investigation with a focus on putting Polya's steps into practice.

INVESTIGATION 1.1d Coin Problem

Variations of this problem are often found in elementary school textbooks because it provides an opportunity to move beyond random guess and test.

If 8 coins total 50 cents, what are the coins?

Solve this problem intentionally using and writing out Polya's four steps of problem solving.

DISCUSSION

Explorations Manual 1.1

STEP 1: UNDERSTAND THE PROBLEM

So often students will jump into a problem without stopping to really understand it. Read a problem more than once before attempting to solve it. Take a few deep breaths and know that you can figure it out with some thinking. Write down the important information and pay attention to what the question is before starting. Here, you have 8 coins, which might be pennies, nickels, dimes, quarters, or half dollars. All together they equal 50 cents. We need to determine what kind of coins we have.

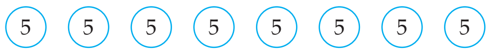
STEP 2: DEVISE A PLAN

There is more than one strategy to solve any problem. Here we could use a diagram, make a table, use reasoning, or use a bag of coins to help us solve it. Let's consider two strategies of making a diagram and using reasoning.

STEP 3: MONITOR THE PLAN

STRATEGY 1 Use a diagram

We could make 8 circles and begin with all nickels: 8 coins = 40¢. What might be the next step?



With a bit of thinking, we can conclude that each time we substitute a dime for a nickel, the total increases by 5 cents. Thus, we need to trade 2 nickels for 2 dimes, and the answer is 6 nickels and 2 dimes.

STRATEGY 2 Use reasoning

Eight nickels would make 40 cents, and 8 dimes would make 80 cents. Because the 8 coins make 50 cents, your first guess will have more nickels than dimes. Even if the guess is wrong—for example, 5 nickels and 3 dimes make 55 cents—you are almost there.

STEP 4: LOOK BACK AT YOUR WORK

We are almost done, but we need to ask questions like: Does my answer make sense? Did I answer the question? Is this the only answer?

We can go back and reread the problem, make sure our solution answers the questions, make sure our answer makes sense, see if we missed any information, and think about alternate ways to get to the solution.

The only other possible solutions are that there might be 1 quarter or 5 pennies. Do you see why? If we make 1 of the coins a quarter, then the other 7 coins must be worth 25 cents. If 5 of those coins are pennies, then we need 2 coins that are worth 20 cents. Aha—2 dimes. Do you see that we could have arrived at the same answer if we had begun with 5 pennies?

Mathematical Bioreactions

Some students have the attitude, "Why do I have to learn this?" Others may run away from learning math. I have heard students say that they read a math question and freeze, not knowing what to do. These are biological reactions called fight, flight, or freeze that get triggered in us when we are faced with something we are afraid of. We have these bioreactions to keep us safe. Whether you have experienced this with math, or something else, the good news is that you have control over this once you understand what is happening. Realizing that math is not going to hurt you, that mistakes are good because they help your brain to grow, and taking deep breaths will all help you to overcome these bioreactions, which is a first step in problem solving and in developing a mathematical growth mindset.

Mathematical Visualization

Many educational psychologists such as Piaget and Bruner have talked about the progression of learning from the concrete to the abstract. This concept of developing ideas through concrete, pictorial, and then abstract (CPA) has helped Singapore's math education system to become one of the best in the world. We will develop many concepts in this book through these CPA steps. Let's look at basic addition and how this progression works.

Concrete: I have 2 pencils and then get 3 more. How many do I have now?

We can lay out 2 pencils and then 3 pencils and easily count 5 pencils together.